ON SOME TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF FILTRATION WITH A LIMITING GRADIENT

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Problems of nonlinear filtration, i.e. of filtration not obeying Darcy's law, are studied chiefly because of their great practical importance. Flows of paraffin-based oils, of oils containing asphaltenes, of oil-water emulsions and even flows of water in mud solutions (oil wells) are non-Newtonian and are characterized by smaller or larger deviations from linearity in the relationship between the rate of filtration $\mathbf{w}$ and the pressure gradient grad p.

The most important of all the nonlinear filtration laws is the, so-called, filtration law with a limiting gradient,

$$\operatorname{grad} p = -\frac{\mu}{k}\mathbf{w} - \gamma\frac{\mathbf{w}}{w} \quad (w > 0), \qquad |\operatorname{grad} p| \leqslant \gamma = \text{const} \quad (w = 0) \qquad (0.1)$$

This law was first used to describe the process of filtration of water through clay [1 and 2] and the filtration of visco-plastic (Bingham type) fluids [3 and 4], but its range of application is much greater.

In particular, it holds within the intermediate region of rates of filtration (not too small, but not large enough to exhibit inertia effects), provided that at high shear rates the relation between tangential stresses and shear rates is almost linear.

With this in mind, we can apply (0.1) to the majority of cases; the region of filtration divides naturally into a region of motion where pressure gradient and filtration rate are connected linearly, and stagnation zones where the fluid is at rest (or moves with negligible velocities if (0.1) holds only approximately). In drilling for oil, stagnation zones represent either the inaccessible regions, or the regions from which oil can only be extracted very slowly. Thus we see that determination of the extent of stagnation zones constitutes a major problem in the theory of nonlinear filtration.

The formation of stagnation zones necessitates a rearrangement of the flow to change the filtration drag and this is only possible in the systems of more than one dimension.

Below we construct a flow pattern and determine the boundaries of stagnation zones for a number of symmetrical configurations of sources and sinks using the Chaplygin transformation ([6], also see [7 and 8]), which was first used in problems of nonlinear filtration by Khristianovich in [9](also see [10]). Rates of filtration in the problem specified above are reduced in the hodograph plane to a solution of boundary value problems for a linear equation in a semistrip with a lengthwise cut. Estimates for solutions and related estimates of the size of the stagnation zone on the physical plane, are obtained. Limit solutions are contructed, which define the character of the flow near the tip of the stagnation zone and make it possible to obtain simple lower estimate of its size.

We should note that related problems were considered in [5], however the form of the law of filtration was changed to simplify it and the problem was subsequently reduced to the solution of Laplace equation in the region with an unknown boundary.

1. Basic equations. Let us consider a plane motion of an incompressible fluid obeying an arbitrary (in general nonlinear) law of filtration which we shall write after Khristianovich [9]

$$\operatorname{grad} H = -\frac{\mathbf{w}}{w}\Phi(w) \qquad \left(H = \frac{k}{\mu}(p + \rho g z)\right) \tag{1.1}$$

Here H is the applied pressure, p is pressure, ρg and μ are specific gravity and viscosity of the fluid, z is the vertical coordinate and k is permeability corresponding to the linear part of the filtration law. Denoting the angle between the x-axis and the filtration rate vector by θ , we can write the equation of motion as

$$\frac{\partial H}{\partial x} = -\Phi(w)\cos\theta, \quad \frac{\partial H}{\partial y} = -\Phi(w)\sin\theta, \frac{\partial(w\cos\theta)}{\partial x} + \frac{\partial(w\sin\theta)}{\partial y} = 0 \tag{1.2}$$

Continuity equation will be identically satisfied, if the following stream function ψ is introduced

$$\frac{\partial\psi}{\partial x} = -w\sin\theta, \qquad \frac{\partial\psi}{\partial y} = w\cos\theta \tag{1.3}$$

As we know [9 and 10], filtration equations become linear if w and θ are taken as independent variables and ψ and H as unknown functions. Then the filtration equations become [9 and 10]

$$\Phi^2\frac{\partial\psi}{\partial w} + w\Phi'(w)\frac{\partial H}{\partial\theta} = 0, \qquad \Phi\frac{\partial\psi}{\partial\theta} - w^2\frac{\partial H}{\partial w} = 0 \tag{1.4}$$

from which we can eliminate H and thus obtain a second order equation for ψ

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{w\Phi(w)}{\Phi'(w)}\frac{\partial^2\psi}{\partial w^2} + \left[2w - \frac{\Phi(w)}{\Phi'(w)} - \frac{w\Phi(w)\Phi''(w)}{[\Phi'(w)]^2}\right]\frac{\partial\psi}{\partial w} = 0 \tag{1.5}$$

The most interesting filtration law is the law with a limiting gradient

$$\Phi(w) = w + \lambda \quad (w > 0), \qquad 0 \leqslant \Phi(w) \leqslant \lambda \qquad (w = 0) \tag{1.6}$$

It is relatively simple and can easily be applied in practice. It presupposes the possibility of formation of stagnation zones or regions, in which $w \equiv 0$. Outside these zones and within the flow, (1.5) becomes

$$\frac{\partial^2\psi}{\partial\theta^2} + w(w+\lambda)\frac{\partial^2\psi}{\partial w^2} + (w-\lambda)\frac{\partial\psi}{\partial w} = 0 \tag{1.7}$$

With the stream function ψ known, we can find H from (1.4), which then assumes the form

$$(w+\lambda)^2\frac{\partial\psi}{\partial w} + w\frac{\partial H}{\partial\theta} = 0, \qquad (w+\lambda)\frac{\partial\psi}{\partial\theta} - w^2\frac{\partial H}{\partial w} = 0 \tag{1.8}$$

Boundary conditions for (1.8) or (1.7) are formulated in the usual manner, the only unusual feature being, that the boundaries of stagnation zones are both, stream lines ψ = const and the lines of zero velocity $w = 0$.

Finally, if ψ and H are determined in terms of w and θ , then a line C on the $w\theta$-plane will be mapped onto the xy-plane of initial independent variables by [9 and 10]

$$x = x_0 - \int_C \frac{\cos\theta}{\Phi(w)}dH + \frac{\sin\theta}{w}d\psi = x_0 - \int_C \frac{\cos\theta}{w+\lambda}dH + \frac{\sin\theta}{w}d\psi$$

$$y = y_0 - \int_C \frac{\sin\theta}{\Phi(w)}dH - \frac{\cos\theta}{w}d\psi = y_0 - \int_C \frac{\sin\theta}{w+\lambda}dH - \frac{\cos\theta}{w}d\psi \tag{1.9}$$

where ψ and H are assumed to be given in terms of w and θ .

2. The mapping of symmetrical regions. 1°. A plane problem of nonlinear filtration can be solved by transformation of a hodograph, provided that the xy-plane can be uniquely mapped onto a sufficiently smooth region of the $w\theta$ hodograph plane. We shall consider several simple problems for which the above statement holds. Consider two sources, each of intensity q at a distance of $2L$ from each other. Then, the flow pattern will be symmetric with respect to the x- and y-axes (Fig. 1) and we shall have at the origin

$$w = 0, \quad |\operatorname{grad} H| = 0 \quad (x = y = 0) \quad (2.1)$$

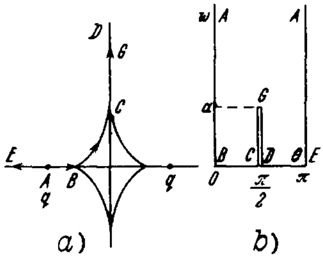


Fig. 1

We shall assume that the functions ψ and H and their first derivatives are continuous up to the boundary of the region of flow. Then we can immediately infer from (2.1), that the origin is surrounded by a stagnation zone within which we have $w = 0$, pressure gradient is tangential to the boundary and its modulus is equal to λ. Hence, the critical points of the boundary (i.e. points at which the stream lines containing the stagnation zone meet), are cusp points (Fig. 1a).

The flow pattern of Fig. 1a is easily mapped onto the hodograph plane (Fig. 1b). Second quadrant of the physical plane becomes a semi-infinite strip with a cut $ABCGDEA$ on the $w\theta$-plane. Length $CG = a$ of the cut is equal to the maximum value of the y-component of the filtration rate achieved at some point G. We cannot exclude a priory the case when the stagnation zone contains the whole y-axis; in this case the point C in Fig. 1a recedes to infinity and the point G in Fig. 1b merges with C and D.

To find ψ, we must solve (1.8) in the region $ABCGDE$ with the following boundary conditions:

$$\psi = 0 \quad \text{on } ABCGD, \qquad \psi = \frac{q}{2} \quad \text{on } EA, \quad \psi = \frac{q}{2}\left(1 - \frac{2(\pi - \theta)}{\pi}\right) \quad \text{on } DE \quad (2.2)$$

Last of these conditions follows from the fact, that at infinity, flow pattern coincides asymptotically with that of a single point source of strength $2q$.

2°. We shall now replace one of the sources of the previous example with a sink of the same strength. This produces an outer stagnation zone (Fig. 2a).

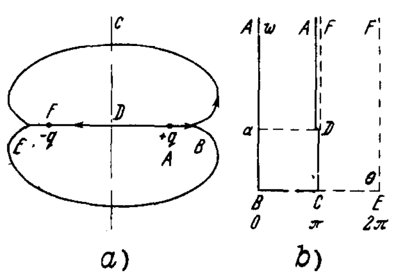


Fig. 2

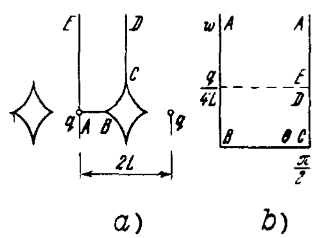


Fig. 3

Indeed, if we assume that the outer stagnation zone is absent, then we must admit the possibility of appearance of a stream line of any length. By the law of filtration (1.6), difference of pressure H between any two points of a stream line cannot be smaller than

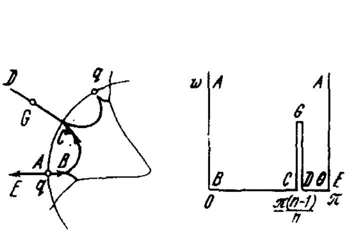
$\lambda \ell$ where ℓ is the length of arc of this stream line. At the same time since the source and the sink can both be surrounded by two closed equipotential lines, pressure drop must be finite for all stream lines contained between these two equipotentials. Since the flow is symmetric with respect to x-axis and antisymmetric with respect to y-axis, the first quadrant of the physical plane again maps onto a semistrip in the hodograph plane $w\theta$ (Fig. 2b). Moreover,

$$\psi = 0 \quad \text{on } ABC, \qquad \psi = {}^1/_2 q \quad \text{on } DA, \qquad H = \text{const} \quad \text{on } CD$$

which, with the second Eq. of (1. 4) taken into account, becomes

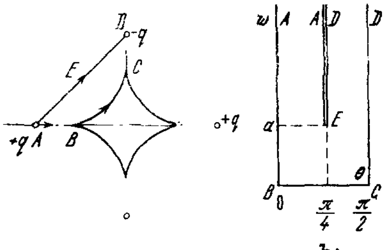
$$\psi = 0 \quad \text{on } ABC, \qquad \psi = {}^1/_2 q \quad \text{on } DA, \qquad \partial\psi / \partial\theta = 0 \quad \text{on } CD \qquad (2.3)$$

The length of the segment $CD = a$ on the hodograph plane must also be determined.



a) b)

Fig. 4



a) b)

Fig. 5

The above two examples are typical both, as qualitative examples of flow patterns in the physicla plane and as associated problems in the hodograph plane. A large number of symmetrical source and sink configurations exist, which can easily be mapped onto the hodograph plane $w\theta$. For example, in the case of an infinite row of sources (Fig. 3a), the basic element $ABCDE$ on the physical plane transforms into a semi-infinite strip (Fig. 3b); here the problem of the stream function is not mixed, $\psi = 0$ on the boundary $ABCD$ and $\psi = {}^1/_4 q$ on EA (Fig. 3b). Similarly we can map two parallel rows of sources or of sources and sinks, circular aggregates of sources and alternating assemblies of sources and sinks encountered in oil-drilling industry in form of doubly periodic meshes (so called five-, seven- and nine-point systems of well distribution). We shall give examples of mapping a n-source circular aggregate (Fig. 4), and an element of a five point system (Fig. 5). In both cases, position of the end of the cut a must be determined separately.

3. Limit cases and estimates for solutions. 1°. We have seen that a number of problems of great practical importance can be reduced to determination of the function ψ satisfying (1. 7) in a semi-infinite strip $0 < \theta < \theta_0 = \text{const}$, $0 < w < \infty$ without a finite (first problem) or an infinite (second problem) segment, which are, respectively $\theta = \theta_1$, $0 < w < a$ and $\theta = \theta_1$, $a < w < \infty$. Here the boundary conditions are

$$\psi = 0 \quad (\theta = 0,\ 0 \leqslant w < \infty;\ w = 0,\ 0 \leqslant \theta \leqslant \theta_1;\ \theta = \theta_1,\ 0 \leqslant w \leqslant a)$$
$$\psi = Q\left(1 - \frac{\theta_0 - \theta}{\theta_0 - \theta_1}\right), \quad (w = 0,\ \theta_1 \leqslant \theta \leqslant \theta_0), \quad \psi = Q\ (\theta = \theta_0,\ 0 \leqslant w < \infty) \qquad (3.1)$$

for the first problem and

$$\psi = 0\,(\theta = 0,\ \theta = \theta_0,\ 0 \leqslant w < \infty;\ w = 0,\ 0 < \theta < \vartheta_0)$$
$$\psi = Q \quad (\theta = \theta_1,\ a \leqslant w < \infty) \tag{3.2}$$

for the second problem.

When the pattern exhibits an additional symmetry in the hodograph plane, then the problem can be reduced to a mixed problem for a narrow semistrip.

It appears that, in general, only approximate solutions can be obtained. Simple estimates can however be given for them, by utilizing the boundary conditions (3.1) or (3.2), together with the fact that the maximum principle holds for the equation (1.7) (see e.g. [11]). By this principle, the function ψ assumes its largest and smallest value on the boundary of the region in which the solution is sought.

We shall now consider the first problem, for the solution of which we have an obvious estimate

$$0 \leqslant \psi\,(w, \theta) \leqslant Q\theta\,/\,\theta_0 \tag{3.3}$$

The right-hand side inequality follows from the fact that $Q\theta/\theta_0$ satisfies (1.7) and assumes values not smaller than ψ at the points of the boundary. Let us now vary the magnitude of the parameter a. Let a and $a' > a$ be two values of a, and let ψ_a and $\psi_{a'}$ be the corresponding solutions. Then in the general domain of definition we have

$$\psi_{a'} \leqslant \psi_a \tag{3.4}$$

In fact, $\psi_{a'}$ and ψ_a coincide at all points belonging to the boundary of definition with exception of the segment $\theta = \theta_1$, $a < w \leqslant a'$ on which $\psi_a \geqslant \psi_{a'} = 0$.

In particular we have

$$\psi_\infty \leqslant \psi_a \leqslant \psi_0 \tag{3.5}$$

where ψ_∞ and ψ_0 denote solutions corresponding to $a = \infty$ and $a = 0$; inequalities (3.4) are, moreover, valid in the general domain of definition of the required solutions.

2°. In the following we shall also need some estimates of behavior of the solution near the boundary of the domain of definition.

Let M be a general point of the boundaries of domains of definition of ψ_a and $\psi_{a'}$, and let P be an interior point. Since by the boundary conditions $\psi_a\,(M) = \psi_{a'}\,(M)$, we have

$$\psi_a\,(P) - \psi_a\,(M) \geqslant \psi_{a'}\,(P) - \psi_{a'}\,(M) \tag{3.6}$$

$$[\psi_a\,(P) - \psi_a\,(M)]\,/\,r_{PM}^{\alpha} \geqslant [\psi_{a'}\,(P) - \psi_{a'}\,(M)]\,/\,r_{PM}^{\alpha} \tag{3.7}$$

Here α is a constant and r_{PM} is the distance between P and M in the $w\theta$-plane. Going in (3.7) to the limit as $P \to M$ we find, that the derivative of ψ_a with respect to r^α at the point on the boundary in any inward direction is not smaller than the corresponding derivative of $\psi_{a'}$. In particular, this is true for the inward normal derivatives. The value of α in (3.7) should be chosen so as to exclude the trivial cases of derivatives becoming zero or infinity, whenever such a choice is feasible.

On the segment of the boundary given by $\theta = 0$, $0 < w < \infty$, direction of the inward normal coincides with the direction of increasing θ, and from (3.7) we have

$$0 \leqslant \partial\psi_{a'}\,/\,\partial\theta \leqslant \partial\psi_a\,/\,\partial\theta \leqslant \partial\psi_0\,/\,\partial\theta \leqslant Q\,/\,\theta_0 \tag{3.8}$$

where we have also utilized the inequalities (3.3) and (3.5).

Similarly, when $\theta = \theta_0$ and directions of increase of the inward normal and of θ are opposite, we have

$$Q\,/\,\theta_0 \leqslant \partial\psi_0\,/\,\partial\theta \leqslant \partial\psi_a\,/\,\partial\theta \leqslant \partial\psi_{a'}\,/\,\partial\theta \leqslant \partial\psi_\infty\,/\,\partial\theta \leqslant Q\,/\,(\theta_0 - \theta_1) \tag{3.9}$$

Further, on the sides of the cut we have

$$\partial\psi_a / \partial\theta \leqslant \partial\psi_{a'} / \partial\theta \leqslant 0 \qquad (3.10)$$

on the left side ($\theta = \theta_1 - 0$, $0 \leq w \leq a$) and

$$Q / (\theta_0 - \theta_1) = \partial\psi_\infty / \partial\theta \leqslant \partial\psi_{a'} / \partial\theta \leqslant \partial\psi_a / \partial\theta \qquad (3.11)$$

on the right side ($\theta = \theta_1 + 0$, $0 \leq w \leq a$).

Let us now follow the behavior of the solution at small values of w within the interval $0 \leq \theta \leq \theta_1$. The solution of (1.7) becoming zero when $w = 0$ and when $\theta = 0$, $\theta = \theta_1$ has, on the segment $0 \leq w < a$, the form

$$\psi = \sum_{m=1}^{\infty} P_m(w) \sin\frac{m\pi\theta}{\theta_1} \qquad (3.12)$$

where $P_m(w)$ ($P_m(0) = 0$) is a solution of the hypergeometric equation

$$w(w + \lambda) y'' + (w - \lambda) y' - (\pi m / \theta_1)^2 y = 0 \qquad (3.13)$$

These solutions have the form (see e.g. [12])

$$P_m(w) = B_m w^2 F(-\alpha_m + 2, \alpha_m + 2, 3, -w/\lambda) \equiv B_m w^2 F_{\alpha_m} \qquad (3.14)$$

where $\alpha_m = \pi m/\theta_1$, F denotes a hypergeometric function and $F_{\alpha_m}(w) = 1$ when $w = 0$. Hence,

$$\psi = w^2 \sum_{m=1}^{\infty} B_m F_{\alpha_m}(w) \sin\frac{\pi m\theta}{\theta_1} \qquad (0 \leqslant \theta \leqslant \theta_1,\ 0 \leqslant w < a) \qquad (3.15)$$

Thus, when $0 \leq \theta < \theta_1$, we have $\psi(\theta, w^2) = 0(w^2)$ $(w \to 0)$ and, assuming that $\alpha = 2$ we obtain from (3.7)

$$0 \leqslant \partial\psi_{a'} / \partial(w^2) \leqslant \partial\psi_a / \partial(w^2) \qquad (0 \leqslant \theta \leqslant \theta_1,\ w = 0) \qquad (3.16)$$

The above argument is not applicable when $a = 0$. However, going in (3.16) indirectly to the limit as $a \to 0$ we obtain

$$0 \leqslant \partial\psi_{a'} / \partial(w^2) \leqslant \partial\psi_a / \partial(w^2) \leqslant \partial\psi_0 / \partial(w^2) \qquad (3.17)$$

$$(0 \leqslant \theta \leqslant \theta_1,\ w = 0)$$

This estimate is not trivial, since $\partial\psi_0 / \partial(w^2)$ becomes infinite only at the point $\theta = \theta_1$.

3°. Let us now examine the results which are obtained when the above estimates are applied to the boundary of the stagnation zone i.e. to the stream line on which the velocity becomes zero. Relations (1.9) cannot be directly used in this case, since $d\psi$ and w also become zero. If, however, we consider the stream lines w = const for small w, then by previous estimate $\psi = 0(w^2)$ along these lines, and

$$\int \frac{\cos\theta}{w} d\psi$$

tends to zero as $w \to 0$. Consequently, for the stream line $\psi = 0$, relations (1.9) become

$$x = x_0 - \int \frac{\cos\theta\, dH}{w + \lambda}, \qquad y = y_0 - \int \frac{\sin\theta\, dH}{w + \lambda} \qquad (3.18)$$

Using (1.8) we can write (3.18) as

$$\begin{aligned} x &= x_0 - \int \frac{\cos\theta}{w^2}\frac{\partial\psi}{\partial\theta} dw - \cos\theta\frac{w + \lambda}{w}\frac{\partial\psi}{\partial w} d\theta \\ y &= y_0 - \int \frac{\sin\theta}{w^2}\frac{\partial\psi}{\partial\theta} dw - \sin\theta\frac{w + \lambda}{w}\frac{\partial\psi}{\partial w} d\theta \end{aligned} \qquad (3.19)$$

Let the source be situated at the point x_A, 0. Then, the nearest point of stagnation zone will have the following coordinates:

$$x_B = x_A + \int_0^\infty \frac{\partial\psi(w,0)}{\partial\theta}\frac{dw}{w^2}, \qquad y_B = 0 \tag{3.20}$$

(since the segment AB has the corresponding semiaxis $0 \le w < \infty$, $\theta = 0$ extending in negative direction in the $w\theta$-plane). Taking into account the inequalities (3.8), we obtain

$$0 \leqslant \int_0^\infty \frac{\partial\psi_{a'}}{\partial\theta}\frac{dw}{w^2} \leqslant \int_0^\infty \frac{\partial\psi_a}{\partial\theta}\frac{dw}{w^2} \leqslant \int_0^\infty \frac{\partial\psi_0}{\partial\theta}\frac{dw}{w^2} \qquad (\theta = 0) \tag{3.21}$$

and

$$x_A \leqslant x_{Ba'} \leqslant x_{Ba} \leqslant x_{B_0} \tag{3.22}$$

Thus, the distance between the source and the stagnation zone increases monotonously with decreasing a; we shall show later that $x_{B_0} < \infty$, hence this distance remains finite as $a \to 0$.

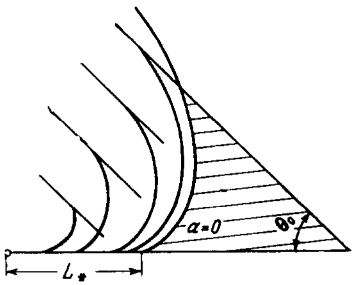


Fig. 6

Let us now consider, on its own, the boundary of stagnation zone corresponding to the segment $w = 0$, $0 \le \theta \le \theta_1$. Here we have

$$x(\theta) - x(0) = \lambda \int_0^\theta \cos\varphi \lim_{w\to 0} \frac{1}{w} \frac{\partial\psi(\varphi, w)}{\partial w} d\varphi$$

$$y(\theta) - y(0) = \lambda \int_0^\theta \sin\varphi \lim_{w\to 0} \frac{1}{w} \frac{\partial\psi(\varphi, w)}{\partial w} d\varphi \tag{3.23}$$

When θ increases, then, by (3.17), x and y increase monotonously up to $\theta = \pi/2$, then y continues to increase, while x begins to decrease. Second relation of (3.23) and (3.17), for $\theta_1 \le \pi$ yield

$$0 \leqslant y_{a'}(\theta) \leqslant y_a(\theta) \leqslant y_0(\theta) \tag{3.24}$$

which together with (3.22) prove, that the corresponding branches of the boundary of stagnation zones corresponding to various values of a, do not intersect.

Indeed, we can easily show that the inequalities (3.21) and (3.22) can be replaced with the strict inequalities, so that, when $a < a'$, then B_a is situated to the right of $B_{a'}$. If, at the same time, the boundaries of stagnation zones intersect, then we have at the point of intersection

$$x_a = x_{a'}, \qquad y_a = y_{a'}, \qquad \theta_a \geqslant \theta_{a'} \tag{3.25}$$

These, however, can easily be shown to be contradictory. Inequalities (3.24) and the monotonous growth of y with increasing θ imply, that the last inequality of (3.25) should be replaced with an equality, and (3.23) demands that $(1/w)\, \partial\psi_a / \partial w \equiv (1/w)\, \partial\psi_{a'} / \partial w$ when $w = 0$ and $0 \leqslant \theta \leqslant \theta_a$. But then the first inequality of (3.23) implies that $x_a - x_{Ba} = x_{a'} - x_{Ba'}$, which, by (3.22), contradicts the first statement of (3.25). Inequalities obtained by us imply, that the distribution of stagnation zones relative to a source changes with decreasing a according to the pattern shown on Fig. 6 illustrating a flow in a wedge-shaped element of angle θ_0. (On the physical plane, planes of symmetry of the flow, play the part of impermeable walls of the wedge). Characteristic distance L representing the distance between the source and an impermeable wall, increases monotonously with decreasing a from zero when $a = \infty$, to infinity when $a = 0$ (the latter part of this statement will be proved later). At the same time, the characteristic distance AB changes from zero to some limit value L_* which is reached at $a = 0$.

Fig. 6 suggests a very simple, although at the same time coarse, method of estimating from below the size of a stagnation zone; if the source intensity Q and the characteristic distance L are given, then the stagnation zone will be larger than the part of the stagnation zone corresponding to $a = 0$ which falls within the wedge (shaded part on Fig. 6). This estimate will improve with increasing L (other condition remaining the same). If on the other hand, $L < L_* \sin\theta_0$, then the estimate becomes trivial. The limiting solution with $a = 0$ describes the flow structure near the cusp of the stagnation zone.

4°. All estimates obtained above referred to the first problem (Subsection 1°), but estimates for solutions of the second problem (conditions (3.2)) can also be obtained by the same simple method. To avoid repetition, we shall formulate them briefly. We obviously have (using the previous notation),

$$0 = \psi_\infty \leqslant \psi_{a'} \leqslant \psi_a \leqslant \psi_0 \qquad (a' > a) \tag{3.26}$$

and solution ψ_0 satisfies, in this case, inequalities

$$0 \leqslant \psi_0(w, \theta) \leqslant Q\theta / \theta_1 \qquad (0 \leqslant \theta \leqslant \theta_1)$$

$$0 \leqslant \psi_0(w, \theta) \leqslant Q(\theta_0 - \theta) / (\theta_0 - \theta_1) \qquad (\theta_1 \leqslant \theta \leqslant \theta_0) \; (0 \leqslant w < \infty) \tag{3.27}$$

Further, we have on the boundary $\theta = 0$, $0 < w < \infty$,

$$0 \leqslant \partial\psi_{a'} / \partial\theta \leqslant \partial\psi_a / \partial\theta \leqslant \partial\psi_0 / \partial\theta \leqslant Q / \theta_1 \tag{3.28}$$

$$-Q / (\theta_0 - \theta_1) \leqslant \partial\psi_0 / \partial\theta \leqslant \partial\psi_a / \partial\theta \leqslant \partial\psi_{a'} / \partial\theta \leqslant 0 \quad \text{for} \quad \theta = \theta_0 \tag{32.9}$$

On the sides of the cut $\theta = \theta_1$, $w > a$, we have

$$Q / \theta_1 \leqslant \partial\psi_0 / \partial\theta \leqslant \partial\psi_a / \partial\theta \leqslant \partial\psi_{a'} / \partial\theta \qquad (\text{on the left side, } \theta = \theta_1 - 0)$$

$$-Q / (\theta_0 - \theta_1) \geqslant \partial\psi_0 / \partial\theta \geqslant \partial\psi_a / \partial\theta \geqslant \partial\psi_{a'} / \partial\theta \quad (\text{on the right side, } \theta = \theta_1 + 0) \tag{3.30}$$

Finally, for small w ($w < a$) we have an expansion analogous to (3.13) (where θ_1 should be replaced with θ_0) and, as before, we obtain the inequality

$$0 \leqslant \partial\psi_{a'} / \partial(w^2) \leqslant \partial\psi_a / \partial(w^2) \leqslant \partial\psi_0 / \partial(w^2) \qquad (0 \leqslant \theta \leqslant \theta_0) \tag{3.31}$$

in which $\partial\psi_0 / \partial(w^2)$ becomes infinite when $\theta = \theta_1$.

5°. As in the first problem, the estimates obtained lead at once to the corresponding estimates for position of the boundaries of the stagnation zone. First, we note that the position of the source relative to the sink is given by

$$x_D - x_A = -\int \frac{\cos\theta_1}{w^2} \frac{\partial\psi(\theta_1, w)}{\partial\theta} dw, \quad y_D - y_A = -\int \frac{\sin\theta_1}{w^2} \frac{\partial\psi(\theta_1, w)}{\partial\theta} dw \tag{3.32}$$

where integration is performed along the sides of the cut in the AED direction. Relations (3.32) give

$$r_{AD} = \left| \int \frac{1}{w^2} \frac{\partial\psi(\theta_1, w)}{\partial\theta} dw \right| \tag{3.33}$$

which, together with (3.31), yields

$$r_{ADa} \geqslant \int_a^\infty \left[\frac{\partial\psi_0(\theta_1 - 0, w)}{\partial\theta} - \frac{\partial\psi_0(\theta_1 + 0, w)}{\partial\theta} \right] \frac{dw}{w^2} \geqslant \left[\frac{Q}{\theta_1} + \frac{Q}{\theta_0 - \theta_1} \right] \frac{1}{a} \tag{3.34}$$

A distance between the source and the cusp of the stagnation zone, is given by

$$0 \leqslant x_{Ba'} - x_A \leqslant x_{Ba} - x_A = \int_0^\infty \frac{\partial \psi_a(0, w)}{\partial \theta} \frac{dw}{w^2} \leqslant x_{B_0} - x_A \qquad (3.35)$$

where, again, the inequalities can easily be represented in a rigorous form.

The argument used to show that the boundaries of stagnation zones corresponding to various a do not intersect, can also be fully repeated. One difficulty arises, namely that the case $\theta_1 \leq \pi$ is now insufficient. If, however, we assume that $\theta_0 \leq 2\pi$ and use the argument given in Subsection 4° for $0 \leq \theta \leq \pi$ repeating it for the segment $\pi \leq \theta \leq \theta_0$ in the direction away from the sink F, then we easily obtain an inequality analogous to (3.35)

$$0 \leqslant x_F - x_{Ea'} \leqslant x_F - x_{Ea} \leqslant x_F - x_{E_0} \qquad (3.36)$$

and the following important theorem.

With Q and λ fixed, width of the region of flow increases monotonously with decreasing a. The distance AB between the source and the cusp of the stagnation zone tends to a finite limit as $a \to 0$. A solution corresponding to $a = 0$ can be used to describe the flow near the point B and rough estimates of minimum size of the stagnation zone can be constructed in the manner indicated in Subsection 3°.

4. Limit solutions. 1°. We shall find now the limit solutions of the first and second problem, corresponding to $a = 0$. Their importance lies in the fact, that they can be written in a sufficiently simple form, using elementary methods. Let us consider the first problem. Putting $a = 0$ we arrive at the following problem: to find a function $\psi(w, \theta)$ satisfying (1.7) in the semistrip $0 \leq \theta \leq \theta_0$, $w \geq 0$, together with conditions

$$\psi(w, 0) = 0, \quad \psi(0, \theta) = 0 \qquad (0 < \theta < \theta_1)$$

$$\psi(0, \theta) = Q\frac{\theta - \theta_1}{\theta_0 - \theta_1} \qquad (\theta_1 < \theta < \theta_0), \qquad \psi(w, \theta_0) = Q \qquad (4.1)$$

Assuming that

$$\psi(w, \theta) = Q\,\theta / \theta_0 + \Psi_0(w, \theta) \qquad (4.2)$$

we find, that, when $\theta = 0$ and $\theta = \theta_0$, ψ_0 becomes zero for all w, while when $w = 0$,

$$\Psi_0(0, \theta) = -Q\,\theta / \theta_0 \qquad (0 < \theta < \theta_1)$$

$$\Psi_0(0, \theta) = -\frac{Q\theta_1}{\theta_0}\frac{\theta_0 - \theta}{\theta_0 - \theta_1} \qquad (\theta_1 < \theta < \theta_0) \qquad (4.3)$$

Expanding the solution into a Fourier series we obtain

$$\Psi_0(w, \theta) = \sum_{m=1}^{\infty} P_{0m}(w) \sin\frac{m\pi\theta}{\theta_0} \qquad (4.4)$$

where $P_{0m}(w)$ are the solutions of

$$w(w + \lambda) \quad Y'' + (w - \lambda)\ Y' - \pi^2 m^2 / \theta_0^2 Y = 0 \qquad (4.5)$$

which become zero as $w \to \infty$ and assume the value

$$P_{0m}(0) = -\frac{2Q\theta_0}{(\theta_0 - \theta_1)\pi^2 m^2} \sin\frac{\pi m\theta_1}{\theta_0} \qquad (4.6)$$

when $w = 0$.

Solution of (4.5) which tends to zero as $w \to \infty$, has the form

$$Y = \left(\frac{\lambda}{w + \lambda}\right)^s F(s - 1, s, 2s + 1, \lambda / (w + \lambda)), \qquad s = \pi m / \theta \qquad (4.7)$$

where F denotes a hypergeometric function. Using a well known integral representation of F (see e.g. [12])

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}}\, dt \tag{4.8}$$

and the condition (4.6), we have

$$P_{0m}(w) = -\frac{2Q}{\theta_0(\theta_0-\theta_1)} \sin\frac{\pi m\theta_1}{\theta_0} \frac{\lambda}{w+\lambda} \frac{s+1}{s} \int_0^1 \frac{t^{s-1}(1-t)^s\, dt}{(w/\lambda+1-t)^{s-1}} \tag{4.9}$$

Inserting (4.9) into (4.4), changing the order of summation and integration and summing a series appearing under the integral sign, we obtain

$$\Psi_0(w, \theta) = -\frac{Q}{\theta_0(\theta_0-\theta_1)} \frac{\lambda}{w+\lambda} \int_0^1 \left(\frac{w+\lambda}{\lambda t} - 1\right) \times$$

$$\times \operatorname{Re}\left\{e^{i\pi(\theta-\theta_1)/\theta_0}\left[\frac{\nu^{\pi/\theta_0}}{1-\nu^{\pi/\theta_0}e^{i\pi(\theta-\theta_0)/\theta_0}} + \int_0^{\nu} \frac{\xi^{(\pi-\theta_0)/\theta_0}\, d\xi}{1-\xi^{\pi/\theta_0}e^{i\pi(\theta-\theta_1)/\theta_0}}\right] - \right. \tag{4.10}$$

$$\left. - e^{i\pi(\theta+\theta_1)/\theta_0}\left[\frac{\nu^{\pi/\theta_0}}{1-\nu^{\pi/\theta_0}e^{i\pi(\theta+\theta_1)/\theta_0}} + \int_0^{\nu} \frac{\xi^{(\pi-\theta_0)/\theta_0}\, d\xi}{1-\xi^{\pi/\theta_0}e^{i\pi(\theta+\theta_1)/\theta_0}}\right]\right\}$$

where

$$\nu = t(1-t)/(w\lambda^{-1}+1-t) \tag{4.11}$$

2°. Let us examine in more detail the solution corresponding to $\theta_0 = 2\theta_1 = \pi$ (the very first problem on two sources). After simple transformations, (4.10) becomes

$$\Psi_0(w, \theta) = -\frac{2Q}{\pi^2} \frac{\lambda}{w+\lambda} \int_0^1 \operatorname{Im}\left\{\frac{2e^{i\theta}\nu}{1+\nu^2e^{2i\theta}} + \right. \tag{4.12}$$

$$\left. + e^{i\theta}\int_0^{\nu} \frac{2d\xi}{1+\xi^2e^{2i\theta}}\right\}\left(\frac{w+\lambda}{\lambda t} - 1\right) dt = -\frac{4Q}{\pi^2} \frac{\lambda}{w+\lambda} \int_0^1 \left(\frac{w+\lambda}{\lambda t} - 1\right) \operatorname{Im} \int_0^{\nu e^{i\theta}} \frac{2d\xi}{(1+\xi^2)^2}$$

The inner integral in (4.12) can be expressed in terms of elementary functions. Let us now establish the position of the stagnation zone corresponding to (4.12), relative to a source. Substituting (4.12) into (3.20) and (4.2), we obtain

$$x_B - x_A = \int_0^{\infty} \left[\frac{Q}{\pi} - \frac{8Q}{\pi^2} \frac{\lambda}{w+\lambda} \int_0^1 \frac{1-t}{(1+\nu^2)^2}\, dt\right] \frac{dw}{w^2} \tag{4.13}$$

It is easy to see that the integrand in (4.13) has a second order zero when $w = 0$ which implies that the integral converges. Integration by parts followed by integration of the inner integral with respect to ν and change of the order of integration, reduces the problem to a single integration which yields

$$x_B - x_A = 0.514\, Q/\lambda = 1.62\, Q/\pi\lambda \tag{4.14}$$

To aid us in investigating the behavior of the boundary of the stagnation zone, we shall now compute $\partial\Psi_0/\partial w$. Differentiation of (4.12) yields

$$\frac{\partial\Psi_0}{\partial w} = \frac{8Q}{\pi^2\lambda} \int_0^1 \frac{(1-t)^2[t-\omega-\omega^2/(1-t)]}{(\omega+1)^2(\omega+1-t)^2} \frac{\sin\theta(1-2\nu^2)-\nu^4\sin 3\theta}{[1+2\nu^2\cos 2\theta+\nu^4]^2}\, dt; \quad \omega = \frac{w}{\lambda} \tag{4.15}$$

Putting $\omega = 0$ we find that the resulting integral is equal to zero, hence $\partial\Psi_0(0,\theta)/\partial w = 0$. From (3.23) we see that the coordinates of the boundary of the stagnation zone are given, respectively, by the real and imaginary part of the following integral:

$$\lambda \int_0^\theta e^{i\varphi}\left(\frac{1}{w}\frac{\partial \Psi_0}{\partial w}\right)_{w=0} d\varphi = \lim_{w\to 0}\frac{\lambda}{w}\int_0^\theta e^{i\varphi}\frac{\partial\psi_0(w,\varphi)}{\partial w}\,d\varphi \equiv \lim_{w\to 0}\frac{\lambda}{w}I_\theta \qquad (4.16)$$

We shall find first I_θ first. Integrating with respect to ν instead of t we can show that the ratio I_θ/w remains finite when $w\to 0$ and $\theta \neq \pi/2$, while

$$\lim_{w\to 0} I_\theta / w = -\frac{4Q}{\pi^2\lambda^2}\int_0^1 (1+t)^2\left[\frac{1}{1+2t^2\cos 2\theta+t^4}-\frac{1}{(1+t^2)^2}+\right.$$
$$\left.+(1+i\,\mathrm{tg}\,\theta)\frac{2t^2\sin^2 2\theta}{(1+2t^2\cos 2\theta+t^4)^2}\right]dt \qquad (4.17)$$

Calculating the integrals in (4.17) and using (3.23), we obtain

$$x(\theta) = x_B + \frac{4Q}{\pi^2\lambda}\left[\frac{\pi}{4}\frac{(1+\sin\theta)^2}{\cos\theta}-\frac{\pi}{4}-\left(\frac{\pi}{2}-\theta\right)\mathrm{tg}\,\theta\right]$$
$$y(\theta) = \frac{4Q}{\pi^2\lambda}\,\mathrm{tg}\,\theta\left[\frac{\pi}{4}\frac{\sin^2\theta}{\cos\theta}+\frac{1}{2}-\frac{\theta\cos 2\theta}{\sin 2\theta}\right] \qquad (4.18)$$

which are the required expressions. We see from them, that the limit stagnation zone is unbounded as $a\to 0$ and consisits of two parabolic branches extending to infinity. On the physical plane this corresponds to a flow about an infinite curvilinear wedge. The boundary is shown on Fig. 7, where the points plotted are accompanied by the values of θ in degrees. The solution obtained is applicable to the problems mentioned previously and it serves a double purpose; first, it describes the flow pattern near the cusp of the stagnation zone, second, it can be used as a zero approximation in deriving solutions corresponding to small a by the method of small parameter.

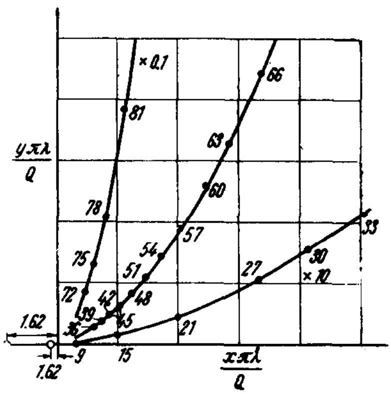


Fig. 7

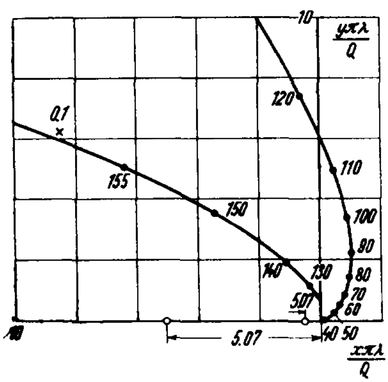


Fig. 8

3°. We shall now construct a limit solution for the second problem. When $a = 0$, the strip $0<\theta<\theta_0$ is split into two parts and solution can be obtained for each part separately, by the same method. Therefore we shall consider a semi-infinite strip $0<w<\infty$, $0<\theta<\theta_1$. The required solution which becomes zero when $\theta = 0$ and $w = 0$ and at Q when $\theta = \theta_1$, has the form

$$\psi(w,\theta) = Q\,\theta/\theta_1 + \Psi^0(w,\theta) \qquad (4.19)$$

where $\Psi^0(w,\theta) = 0$ along the sides of the strip and

$$\Psi^\circ(0, \theta) = -Q\theta/\theta_1 \tag{4.20}$$

The solution can be expanded into a Fourier series

$$\Psi^\circ(w, \theta) = \sum_{m=1}^{\infty} P_m{}^\circ(w) \sin\frac{\pi m\theta}{\theta_1} \tag{4.21}$$

where $P^\circ_m(w)$ are found in exactly the same manner as $P_{m0}(w)$ for the first problem (relations (4. 4) to (4. 9)) and are

$$P_m{}^\circ(w) = (-1)^m \frac{2Q}{\theta_1}\frac{\lambda}{w+\lambda}(p+1)\int_0^1 t^{p-1}(1-t)^p(w/\lambda+1-t)^{1-p}dt \tag{4.22}$$

$$(p = \pi m/\theta_1)$$

Inserting (4. 22) into (4. 21), interchanging the summation with integration and performing the summation of the series under the integral sign, we obtain

$$\Psi^\circ(w, \theta) = -\frac{2Q}{\theta_1}\frac{\lambda}{w+\lambda}\,\mathrm{Im}\int_0^1 \frac{w/\lambda+1-t}{t}\,\frac{\nu^{\pi/\theta_1}e^{i\theta(\theta-\pi)/\theta_1^2}+\nu^{2\pi/\theta_1}e^{2i\pi\theta/\theta_1}}{(1+\nu^{\pi/\theta_1}e^{i\pi\theta/\theta_1})}\,dt \tag{4.23}$$

where ν is given by (4. 11).

Let us now consider the particular case $\theta_1 = \pi$ corresponding to the source-sink pair of equal intensity. Taking into account (4. 23), we have

$$\psi = \frac{Q\theta}{\pi} + \Psi^\circ(w, \theta) = \frac{Q\theta}{\pi} - \frac{2Q}{\pi}\frac{\lambda}{w+\lambda}\,\mathrm{Im}\int_0^1 \frac{(1-t)(2+\nu e^{i\theta})e^{i\theta}dt}{(1+\nu e^{i\theta})^2} \tag{4.24}$$

Derivative $\partial\psi/\partial\theta$ is given by

$$\frac{\partial\psi}{\partial\theta} = \frac{Q}{\pi} - \frac{4Q}{\pi}\,\mathrm{Re}\,\frac{\lambda e^{i\theta}}{w+\lambda}\int_0^1 \frac{(1-t)\,dt}{(1+\nu e^{i\theta})^3} \tag{4.25}$$

Hence, the distance between the source and the cusp of the stagnation zone is given by

$$x_B - x_A = \frac{4Q}{\pi\lambda}\int_0^\infty \frac{d\omega}{\omega(\omega+1)^2}\int_0^1 \frac{(1-t)[1-2\nu-t-\nu t-\omega(2\nu-1)]}{(1+\nu)^4(\omega+1-t)}\,dt \tag{4.26}$$

which on integration with respect to ν and change of the order of integration, gives

$$x_B - x_A = 1.61\,Q/\lambda = 5.07\,Q/\pi\lambda \tag{4.27}$$

To find the form of the stagnation zone boundary, we shall differentiate (4. 24) twice with respect to w and pass to the limit as $w \to 0$. This gives

$$\chi(\theta) = \lim_{w\to 0}\frac{1}{w}\frac{\partial\Psi^\circ}{\partial w} = \frac{24Q}{\pi\lambda^2}\int_0^1 \frac{t(1+t)[t+\cos\theta(1+t^2)]\sin\theta\,dt}{(1+2t\cos\theta+t^2)^4} \tag{4.28}$$

Let us now compute the integral

$$I_\varphi = \int_0^\varphi e^{i\theta}\chi(\theta)\,d\theta = \frac{Q}{2\pi\lambda^2}\left\{\frac{3\cos\varphi(1-\cos\varphi)}{\sin^2\varphi} - \frac{3}{2} + \right.$$
$$+\frac{(1-\cos\varphi)^2}{\sin^2\varphi}\left[2\cos\varphi+2+\frac{\varphi(1+2\cos\varphi)}{\sin\varphi}\right] +$$
$$\left. + i\left[\frac{1-\cos\varphi}{\sin\varphi}\left(1-\frac{\varphi}{\sin\varphi}\right)+\frac{2(1-\cos\varphi)^2}{\sin\varphi}\left(1+\frac{\varphi}{\sin\varphi}\right)\right]\right\} \tag{4.29}$$

From (4.29) it follows, in particular, that

$$I_\varphi = \frac{Q}{2\pi\lambda^2}\left(\varphi^2 + i\,\frac{11}{12}\,\varphi^3\right)(1 + O(\varphi^2)) \quad \text{when } \varphi \to 0$$

$$I_\varphi = \frac{2Q}{\lambda^2}\left(-\frac{1}{\sin^3\varphi} + i\,\frac{3}{2\sin^2\varphi}\right)(1 + O(\sin\varphi)) \quad \text{when } \varphi \to \pi \tag{4.30}$$

Separating the real and imaginary parts of I_φ, we obtain the coordinates of the points of the stagantion zone

$$x(\varphi) = x_B + \lambda\,\mathrm{Re}\,I_\varphi, \quad y(\varphi) = \lambda\,\mathrm{Im}\,I_\varphi \tag{4.31}$$

Fig. 8 shows the stagnation zone boundary corresponding to (4.29) (where the numbers denote the values of φ in degrees). The solution obtained is asymptotically similar to that obtained previously for the first problem.

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